A priori estimates for the complex Hessian equations

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Abstract

We prove some L^∞ a priori estimates as well as existence and stability theorems for the weak solutions of the complex Hessian equations in domains of $\mathbb{C}^n$ and on compact Kähler manifolds. We also show optimal L^p integrability for m-subharmonic functions with compact singularities, thus partially confirming a conjecture of Błocki. Finally we obtain a local regularity result for $W^{2,p}$ solutions of the real and complex Hessian equations under suitable regularity assumptions on the right hand side. In the real case the method of this proof improves a result of Urbas.

Introduction

Hessian equations. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ be the set of eigenvalues of a Hermitian $n \times n$ matrix A. By $S_m(A)$ denote the m-th elementary symmetric function of λ:

$$S_m(A) = \sum_{0<j_1<...<j_m\leq n} \lambda_{j_1}\,\lambda_{j_2}\,...\,\lambda_{j_m}\,.$$

If A is the complex Hessian of a real valued C^2 function u defined in $\Omega \subset \mathbb{C}^n$ then we have a pointwise defined function

$$\sigma_m(u_{z_j\bar{z}_k})(z) = S_m\big((u_{z_j\bar{z}_k}(z))\big).$$

In terms of differential forms, with $d = \partial + \bar{\partial},\ d^c = i(\bar{\partial} - \partial)$ and $\beta = dd^c||z||^2$ this function satisfies

$$(dd^c u)^m \wedge \beta^{n-m} = \frac{m!(n-m)!}{n!}\sigma_m(u_{z_j\bar{z}_k})\beta^n.$$

We call a C^2 function $u : \Omega \to \mathbb{C}^n$ *m-subharmonic (m-sh)* if the forms

$$(dd^c u)^k \wedge \beta^{n-k}$$

are positive for $k = 1, ..., m$ (in particular u is subharmonic). If u is subharmonic but not smooth then one can definine m-sh function via inequalities for currents (see definitions in Preliminaries).

As shown by Błocki in [Bl1] m-sh functions are the right class of admissible solutions to the complex Hessian equation

$$(dd^c u)^m \wedge \beta^{n-m} = f\beta^n \tag{0.1}$$

for given nonnegative function f. Observe that for $m = 1$ this is the Poisson equation and for $m = n$ the complex Monge-Ampère equation.

In analogy to the above one can define *m-subharmonic functions with respect to a Kähler form* ω *($m - \omega$-sh)* and the corresponding Hessian equation just replacing β with ω in the preceeding definitions. This definition can also be extended to subharmonic functions. Then one can consider such functions on Kähler manifolds.

Since on compact Kähler manifolds the sets of $m - \omega$-sh functions are trivial we define in this case $\omega - m$-subharmonic ($\omega - m$-sh) functions requiring that

$$(dd^c u + \omega)^k \wedge \omega^{n-k} \geq 0, \quad k = 1, ..., m.$$

and consider the Hessian equation on a compact Kähler manifold X, as in [Hou] and [HMW]:

$$(dd^c u + \omega)^m \wedge \omega^{n-m} = f\omega^n, \quad \int_X f\omega^n = \int_X \omega^n. \tag{0.2}$$

Solving the equation we look for u which is $\omega - m$-sh. The normalization of f is necessary because of the Stokes theorem and the Kähler condition $d\omega = 0$.

Background. The real Hessian equation was studied in many papers, for example in [CNS], [ITW], [Kr], [Tr1], [TW], [La], [CW], [U]; to mention only a few. In particular the Dirichlet problem is solvable for smooth and strictly positive right hand side under natural convexity assumptions on the boundary of the considered domain ([CNS]). This result is the starting point of study of degenerate Hessian equations ([ITW]) and regularity of weak solutions ([U]). Furthermore a non linear potential theory has been developed ([TW], [La]). We refer to [W1] for state-of-the-art survey of the real Hessian equation theory. It is somewhat interesting that real and complex theories are very different, and attempts to use directly the "real" methods to the complex Hessian equation often fail. See [Bl2] or [Bl3] for a detailed study of those discrepancies.

The complex Hessian equation (0.1) in domains of $\mathbb{C}^n$ was first considered by S.Y. Li [Li]. His main result says that if Ω is smoothly bounded and $(m-1)$-pseudoconvex (that means that $S_j, \ j = 1, \cdots, m-1$ applied to the Levi form of $\partial\Omega$ are positive on the complex tangent to $\partial\Omega$) then, for smooth boundary data and for smooth, positive right hand side there exists a unique smooth solution of the Dirichlet problem for the Hessian equation. The proof is in the spirit of the one in [CNS].

In [Bl1] Błocki considered also weak solutions of the equation, for possibly degenarate right hand side, introducing some elements of potential theory for m-sh functions based on positivity of currents which are used in the definition. He proved that the m-sh function u is maximal in this class if and only if

$$(dd^c u)^m \wedge \beta^{n-m} = 0.$$

Furthermore he described the maximal domain of definition of the Hessian operator.

As for the equation on compact Kähler manifolds (0.2) Hou [Hou] has shown that the solutions, for smooth positive f, exist under the assumption that the metric has nonnegative holomorphic bisectional curvature. Similar results were

independently obtained in [Ko], [J]. Despite the further efforts [HMW] the general case is still open.

New results. The m-subharmonic functions for $m < n$ are much more difficult to handle than the plurisubharmonic ones ($m = n$). They lack a nice geometric description by the mean value property along planes, there is no invariance of the family under holomorphic mappings, and so forth. The cones of m-ω-sh functions are even worse - they are not invariant under translations. Despite that the pluripotential theory methods developed in [BT2], [K1], [K2], [K4] for the Monge-Ampère equation can be adapted to the Hessian equations. The crucial estimate between volume and capacity in Proposition 2.1 allowed us to prove a sharp integrability statement (conjectured in a stronger form in [Bl1]): m-subharmonic functions, $m < n$, belong to L^q for any $q < \frac{mn}{n-m}$, if their level sets are relatively compact in the domain where they are defined. For a plurisubharmonic function u much stronger statement is true: $\exp(-au)$ is locally integrable for some $c > 0$. This accounts for the difference in statements of L^∞ estimates for the Hessian equations and the Monge-Ampère equation. We show a priori bounds L^∞ for the solutions of

$$(dd^c u)^m \wedge \beta^{n-m} = f\omega^n \tag{0.3}$$

(with continuous boundary data) and those of (0.2) with f belonging to L^q, $q < \frac{n}{n-m}$. We also get strong stability theorems for those solutions. As a consequence one obtains that the families of solutions corresponding to data uniformly bounded in L^q norms are equicontinuous.

The a priori estimates lead to the (continuous) solution of the Dirichlet problem in $(m-1)$-pseudoconvex domains for nonnegative right hand side in the same L^q spaces as above (Theorem 2.10). The corresponding existence result is also true on compact Kähler manifolds with nonnegative holomorphic bisectional curvature (Theorem 3.3). Those are the extensions of theorems in [Li] and [Hou]. Finally we prove the local regularity statement in Theorem 4.1 which in the case of the Monge-Ampère equation is due to Błocki and Dinew [BD]. It is worth noting that our methods applied to the real Hessian equations yield improvement of the regularity exponent obtained by Urbas ([U]).

1 Preliminaries

We briefly recall the notions that we shall need later on. We start with a linear algebra toolkit.

Linear algebra preliminaries. Consider the set $\mathcal{M}_n$ of all Hermitian symmetric $n \times n$ matrices. For a given matrix $M \in \mathcal{M}_n$ let $\lambda(M) = (\lambda_1, \lambda_2, ..., \lambda_n)$ be its eigenvalues arranged in the decreasing order and let

$$S_k(M) = S_k(\lambda(M)) = \sum_{0<j_1<...<j_m\leq n} \lambda_{j_1} \lambda_{j_2} ... \lambda_{j_m}$$

be the k-th elementary symmetric polynomial applied to the vector $\lambda(M)$.

Then one can define the positive cones Γ_m as follows

$$\Gamma_m = \{\lambda \in \mathbb{R}^n | \ S_1(\lambda) > 0, \ \cdots, \ S_m(\lambda) > 0\}. \tag{1.1}$$

Note that the definition of Γ_m is non linear if $m > 1$ hence a priori it is unclear whether these sets are indeed convex cones. But the vectors in Γ_m, and hence the set of matrices with corresponding eigenvalues enjoy several convexity properties resembling the properties of positive definite matrices, and in partiular the convexity of Γ_m.

Let now V be a fixed positive definite Hermitian matrix and $\lambda_i(V)$ be the eigenvalues of a Hermitian matrix M with respect to V. The we can analogously define the sets $\Gamma_k(V)$.

Below we list the properties of these cones that will be used later on:

1. (Maclaurin's inequality) If $\lambda \in \Gamma_m$ then $(\frac{S_j}{\binom{n}{j}})^{\frac{1}{j}} \geq (\frac{S_i}{\binom{n}{i}})^{\frac{1}{i}}$ for $1 \leq j \leq i \leq m$;
2. (Gårding's inequality, [Ga]) Γ_m is a convex cone for any m and the function $S_m^{\frac{1}{m}}$ is concave when restriced to Γ_m;
3. ([W1]) Let $S_{k;i}(\lambda) := S_k(\lambda)_{\lambda_i=0} = \frac{\partial S_{k+1}}{\partial \lambda_i}(\lambda)$. Then for any $\lambda,\ \mu \in \Gamma_m$
$$\sum_{i=1}^{n} \mu_i S_{m-1;i}(\lambda) \geq m S_m(\mu)^{\frac{1}{m}} S_m(\lambda)^{\frac{m-1}{m}}.$$

We refer to [Bl1] or [W1] for further properties of these cones.

Potential theoretic aspects of m-subharmonic functions. Let us fix a relatively compact domain $\Omega \Subset \mathbb{C}^n$. Let also $d = \partial + \bar{\partial}$ and $d^c := i(\bar{\partial} - \partial)$ be the standard exterior differentiation operators. By $\beta := dd^c||z||^2$ we denote the Euclidean Kähler form in $\mathbb{C}^n$.

Given a $\mathcal{C}^2(\Omega)$ function u we call it $m - \beta$-subharmonic if for any $z \in \Omega$ the Hessian matrix $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(z)$ has eigenvalues forming a vector in the closure of the cone Γ_m. Analogously if ω is any other Kähler form in Ω, u is $m{-}\omega$-subharmonic if the Hessian matrix has eigenvalues at z forming a vector in $\bar{\Gamma}_m(\omega(z))$ (the latter set will depend on z in general).

Since the $\omega = \beta$ is the most natural case in the flat domains we shall call $m - \beta$-subharmonic functions just m-subharmonic or m-sh for short.

Observe that in the language of differential forms u is $m - \omega$-subharmonic if and only if the following inequalities hold:

$$(dd^c u)^k \wedge \omega^{n-k} \geq 0,\ k = 1,\ \cdots,\ m.$$

It was obsreved by Błocki ([Bl1]) that, following the ideas of Bedford and Taylor ([BT1], [BT2]), one can relax the smoothness requirement on u and develop a non linear version of potential theory for Hessian operators.

The relevant definitions are as follows:

Definition 1.1. *Let u be a subharmonic function on a domain $\Omega \Subset \mathbb{C}^n$. Then u is called m-subharmonic (m-sh for short) if for any collection of $\mathcal{C}^2$-smooth m-sh functions $v_1,\ \cdots,\ v_{m-1}$ the inequality*

$$dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{m-1} \wedge \beta^{n-m} \geq 0$$

holds in the weak sense of currents. For a general Kähler form ω the notion of $m - \omega$-subharmonic function is defined by formally stronger condition: locally,

in a neighbourhood of any given point, there exists a decreasing to u sequence of $\mathcal{C}^2$-smooth $m-\omega$-sh functions u_j such that for any set of $\mathcal{C}^2$-smooth $m-\omega$-sh functions $v_1, \cdots, v_{m-1}$ the inequality

$$dd^c u_j \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{m-1} \wedge \omega^{n-m} \geq 0$$

is satisfied. (For $\omega = \beta$ this condition is satisfied due to part 4 of Proposition 1.3 below.)

The set of all $m-\omega$-sh functions is denoted by $\mathcal{SH}_m(\omega, \Omega)$.

Remark 1.2. *It is enough to test m-subharmonicity of u against a collection of m-sh quadratic polynomials (see [Bl1]).*

Using the approximating sequence u_j from the definition one can follow the Bedford and Taylor construction from [BT2] of the wedge products of currents given by locally bounded $m-\omega$-sh functions. They are defined inductively by

$$dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge \omega^{n-m} := dd^c(u_1 \wedge \cdots \wedge dd^c u_p \wedge \omega^{n-m}).$$

It can be shown (see [Bl1]) that analogously to the pluripotential setting these currents are continuous under monotone or uniform convergence of their potentials.

Here we list some basic facts about m-subharmonicity (assuming $\mathcal{C}^2$ smoothness).

Proposition 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Then*

1. $\mathcal{SH}_1(\omega, \Omega) \subset \mathcal{SH}_2(\omega, \Omega) \subset \cdots \subset \mathcal{SH}_n(\omega, \Omega)$,
2. $\mathcal{SH}_m(\omega, \Omega)$ *is a convex cone,*
3. *If $u \in \mathcal{SH}_m(\omega, \Omega)$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^2$-smooth convex, increasing function then $\gamma \circ u \in \mathcal{SH}_m(\omega, \Omega)$,*
4. *the standard regularizations $u * \rho_\varepsilon$ of a m-sh function is again m-sh.*

Proof. The first claim is trivial. Second claim is proved in [Bl1], with the use of Gårding's inequality [Ga]. Last two claims are more or less standard and their proofs are analogous to corresponding results for $\mathcal{PSH}$ functions. Observe that the last property does fail for a general Kähler form ω. □

The following two theorems, known as comparison principles in pluripotential theory, follow essentially from the same arguments as in the case $m = n$:

Theorem 1.4. *Let u, v be continuous $m-\omega$-sh functions in a domain $\Omega \in \mathbb{C}^n$. Suppose that $\liminf_{z\to\partial\Omega}(u-v)(z) \geq 0$ then*

$$\int_{\{u<v\}} (dd^c v)^m \wedge \omega^{n-m} \leq \int_{\{u<v\}} (dd^c u)^m \wedge \omega^{n-m}.$$

Theorem 1.5. *Let u, v be continuous $m-\omega$-sh functions in a domain $\Omega \in \mathbb{C}^n$. Suppose that $\liminf_{z\to\partial\Omega}(u-v)(z) \geq 0$ and $(dd^c v)^m \wedge \omega^{n-m} \geq (dd^c u)^m \wedge \omega^{n-m}$. Then $v \leq u$ in Ω.*

The last result yields, in particular, uniqueness of bounded weak solutions of the Dirichlet problem. As for the existence we have the following fundamental existence theorem due to S. Y. Li ([Li]):

Theorem 1.6. *Let Ω be a smoothly bounded relatively compact domain in $\mathbb{C}^n$. Suppose that $\partial\Omega$ is $(m-1)$-pseudoconvex (that means that Levi form at any point $p \in \partial\Omega$ has its eigenvalues in the cone Γ_{m-1}). Let φ be a smooth function on $\partial\Omega$ and f a strictly positive and smooth function in Ω. Then the Dirichlet problem*

$$\begin{cases} u \in \mathcal{SH}_m(\Omega, \beta) \cap \mathcal{C}(\bar{\Omega}); \\ (dd^c u)^m \wedge \beta^{n-m} = f \\ u|_{\partial\Omega} = \varphi \end{cases}$$

has a smooth solution u.

Finally let us mention that convexity properties of the cones Γ_m yield the following mixed Hessian inequalities:

Proposition 1.7. *Let $u_1, \cdots, u_m$ be m-sh $\mathcal{C}^2$ functions in some domain $\Omega \in \mathbb{C}^n$. Suppose $(dd^c u_j)^m \wedge \beta^{n-m} = f_j$ for some continuous non negative functions f_j. Then*

$$dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m} \geq (f_1 \cdots f_m)^{\frac{1}{m}} \beta^n.$$

Proof. Pointwise this reduces to the Gårding inequality; see also inequality 3. above for the case $u_2 = u_3 = \cdots = u_m$. □

Later on in Theorem 2.12 we shall see that the smoothness assumptions here can be considerably relaxed.

Kähler setting. Given a compact Kähler manifold (X, ω) we can define the cones $\mathcal{SH}_m(X, \omega)$ of those functions u for which, in a local chart Ω where ω has a potential ρ, the function $u + \rho$ belongs to $\mathcal{SH}_m(\Omega, \omega)$. The definition is independent of the choice of the chart and the potential. This essentially allows to carry over all local reults to this setting. We refer to [K5] for the plurisubharmonic $(m = n)$ case.

The comparison principle on compact manifolds reads as follows:

Proposition 1.8. *Let (X, ω) be a compact Kähler manifold and u, v be continuous functions in $\mathcal{SH}_m(X, \omega)$. Then*

$$\int_{\{u<v\}} (\omega + dd^c v)^m \wedge \omega^{n-m} \leq \int_{\{u<v\}} (\omega + dd^c u)^m \wedge \omega^{n-m}.$$

Proof. One can repeat the proof for psh functions from [K4] or [K5]. □

Observe that the cones $\Gamma_k(\omega)$ are not fixed but according to an observation of Hou ([Hou]) these are invariant under the parallel transport defined by the Levi-Civita connection associated to ω.

2 L^∞ estimates and existence of weak solutions in domains

In this section we state the results for $0 < m < n$. Let us denote by $B(a,r)$ the ball in $\mathbb{C}^n$ with center a and radius r. Let also ω be a Kähler form defined in a neighbourhood of the closure of a set Ω considered below and $V = \omega^n$ be the volume form associated to ω.

Let $\mathcal{SH}_m(\omega, \overline{\Omega})$ denote the class of $m-\omega$-sh functions which are continuous in $\overline{\Omega}$.

Proposition 2.1. *For $p < \frac{n}{n-m}$ and an open set $\Omega \subset B(0,1) = B$ there exists $C(p)$ such that for any $K \subset\subset \Omega$,*

$$V(K) \leq C(p) cap_m^p(K, \Omega),$$

where

$$cap_m(K, \Omega) = \sup\{ \int_K (dd^c u)^m \wedge \omega^{n-m}, \ u \in \mathcal{SH}_m(\omega, \overline{\Omega}), 0 \leq u \leq 1\}.$$

Proof. If $V(K) = 0$ then the inequality trivially holds. Assume from now on that $V(K) > 0$. Fix any $\epsilon \in (0, 1/2)$ and set $f = [V(K)]^{2\epsilon-1}\chi_K$, where χ_K denotes the characteristic function of the set K. Solve the complex Monge-Ampère equation in B to find $v \in PSH_\omega(B) \cap C(\overline{\Omega})$ with $v = 0$ on ∂B and

$$(dd^c v)^n = f\omega^n.$$

By the inequality between mixed Monge-Ampère measures (see [K5], [D])

$$(dd^c v)^m \wedge \omega^{n-m} \geq [V(K)]^{(2\epsilon-1)\frac{m}{n}} \chi_K \omega^n. \tag{2.1}$$

For $q = 1 + \epsilon$

$$\int_B f^q dV = [V(K)]^{(2\epsilon-1)(1+\epsilon)+1} = [V(K)]^{\epsilon+2\epsilon^2} \leq V(B).$$

So, by [K1], there exists $c > 0$, independent of K (though dependent on ϵ), such that $||v|| \leq 1/c$. Take $u = cv$. Then, using (2.1)

$$cap_m(K, \Omega) \geq \int_K (dd^c u)^m \wedge \omega^{n-m} \geq c^m [V(K)]^{(2\epsilon-1)\frac{m}{n}+1}.$$

Therefore

$$V(K) \leq C cap_m^{\frac{n}{n-m+2m\epsilon}}(K, \Omega),$$

which proves the claim.

□

Proposition 2.2. *Let Ω and p be as above and consider $u \in \mathcal{SH}_m(\omega, \overline{\Omega})$ with $u = 0$ on $\partial\Omega$ and*

$$\int_\Omega (dd^c u)^m \wedge \omega^{n-m} \leq 1.$$

Then for $U(s) = \{u < -s\}$ we have

$$cap_m(U(s), \Omega) \leq s^{-m}$$

and

$$V_{2n}(U(s)) \leq C(p)s^{-pm}.$$

In particular $u \in L^q(\Omega)$ *for any* $q < \frac{mn}{n-m}$, *and this remains true whenever* u *is bounded in some neighborhood of the boundary of* Ω.

Proof. Fix $\epsilon > 0, t > 1$ and $K \subset U(s)$ and find $v \in \mathcal{SH}_m(\omega, \overline{\Omega})$ with $-1 \leq v \leq 0$ and

$$\int_K (dd^c v)^m \wedge \omega^{n-m} \geq cap_m(K, \Omega) - \epsilon.$$

Then, using the comparison principle [BT1], [Bl1]

$$cap_m(K, \Omega) - \epsilon \leq \int_K (dd^c v)^m \wedge \omega^{n-m}$$
$$\leq \int_{\{-\frac{t}{s}u<v\}} (dd^c v)^m \wedge \omega^{n-m} \leq (\frac{t}{s})^m \int_\Omega (dd^c u)^m \wedge \omega^{n-m} \leq (\frac{t}{s})^m.$$

To finish the proof of the first estimate recall that $cap_m(U(s), \Omega)$ is the supremum of $cap_m(K, \Omega)$ over compact $K \subset U(s)$ and let $\epsilon \to 0$ and $t \to 1$. Then the estimate of the volume follows from Proposition 2.1.

□

Remark 2.3. *The bound for* q *above is optimal as the function*

$$G(z) = -|z|^{2-2n/m}$$

is m-sh and belongs to L^q_{loc} *if and only if* $q < \frac{mn}{n-m}$.

In [Bl1] Błocki conjuctured that any m-sh function belongs to $L^q_{loc}(\Omega)$ for any $q < \frac{mn}{n-m}$. He proved this for $q < \frac{n}{n-m}$. The above proposition confirms partially the conjecture - under the extra assumption of boudedness near the boundary. Still the question about the local inegrability remains open.

We now proceed to proving the L^∞ a priori estimates for the Hessian equation with the right hand side controlled in terms of the capacity.

Lemma 2.4. *For* $p \in (1, \frac{n}{n-m})$ *and an open set* $\Omega \subset B$ *consider* $u, v \in \mathcal{SH}_m(\omega, \overline{\Omega})$ *satisfying*

$$\int_K (dd^c u)^m \wedge \omega^{n-m} \leq A cap_m^p(K, \Omega)$$

for some $A > 0$ *and any compact* $K \subset \Omega$. *If the sets* $U(s) = \{u - s < v\}$ *are nonempty and relatively compact in* Ω *for* $s \in (s_0, s_0 + t_0)$ *then there exists a constant* $C(p, A)$ *such that*

$$t_0 \leq C(p, A) cap_m^{p/n}(U(s_0 + t_0), \Omega).$$

Proof. Using the notation

$$a(s) = cap_m(U(s), \Omega), \quad b(s) = \int_{U(s)} (dd^c u)^m \wedge \omega^{n-m}$$

we claim that

$$t^m a(s) \leq b(s + t), \quad t \in (0, s_0 + t_0 - s). \tag{2.2}$$

Indeed, for fixed compact $K \subset U(s)$ take $w_1 \in \mathcal{SH}_m(\omega, \overline{\Omega})$, $-1 \leq w_1 \leq 0$ such that

$$\int_K (dd^c w_1)^m \wedge \omega^{n-m} \geq cap_m(K, \Omega) - \epsilon.$$

Then for $w_2 = \frac{1}{t}(u - s - t)$ one readily verifies that $K \subset V \subset U(s+t)$, where $V = \{w_2 < w_1 + \frac{1}{t}v\}$. So, by the comparison principle

$$cap_m(K, \Omega) - \epsilon \leq \int_K (dd^c(w_1 + \frac{1}{t}v))^m \wedge \omega^{n-m}$$
$$\leq \int_V (dd^c(w_1 + \frac{1}{t}v))^m \wedge \omega^{n-m} \leq \int_V (dd^c w_2)^m \wedge \omega^{n-m} \leq t^{-m} b(s+t).$$

Having (2.2) one proceeds as in the proof of Lemma 4.3 in [K3] (with $h(x) = x^{m(p-1)}$) to reach the conclusion.

□

Coupling this with the volume estimate in Proposition 2.1 we obtain a priori estimates for the solutions of Hessian equations with the right hand side in some L^q spaces.

Theorem 2.5. *Take $q > n/m$. Then the conjugate q' of q satisfies $q' < n/(n-m)$. Fix $p' \in (q', n/(n-m))$ and $p = p'/q' > 1$. Consider $u, v \in \mathcal{SH}_m(\omega, \overline{\Omega})$ such that $u \geq v$ on $\partial\Omega$, $\{u < v\} \neq \emptyset$ and*

$$(dd^c u)^m \wedge \omega^{n-m} = f\omega^n$$

for some $f \in L^q(\Omega, dV)$. Then

$$\sup(v - u) \leq c(p', q, ||f||_{L^q(\Omega)}) ||(v-u)_+||_{L^{q'}(\Omega)}^{\frac{p}{n+p(m+1)}}, \quad (v-u)_+ := \max(v - u, 0).$$

Proof. By the Hölder inequality and Proposition 2.1, for a compact set $K \subset \Omega$ we have

$$\int_K f\omega^n \leq ||f||_q V(K)^{1/q'} \leq C(p) ||f||_{L^q(\Omega)} cap_m^p(K, \Omega).$$

Therefore, by Lemma 2.4, we get for $t = \frac{1}{2}\sup(v - u)$ and $E(t) = \{u + t < v\}$

$$t \leq c(p', q, ||f||_{L^q(\Omega)}) cap_m^{p/n}(E(t), \Omega). \tag{2.3}$$

To shorten notation set $a(t) = cap_m(E(2t), \Omega)$. Take $w \in \mathcal{SH}_m(\omega, \overline{\Omega})$, $-1 \leq w \leq 0$ such that

$$\int_{E(2t)} (dd^c w)^m \wedge \omega^{n-m} \geq \frac{1}{2} a(t).$$

Observe that for $V = \{u < tw + v - t\}$ the following inclusions hold

$$E(2t) \subset V \subset E(t).$$

Applying the comparison principle we thus get

$$\frac{1}{2} a(t) t^m \leq \int_{E(2t)} [dd^c(tw + v)]^m \omega^{n-m} \leq \int_V (dd^c u)^m \wedge \omega^{n-m}$$
$$\leq \int_{E(t)} f \, dV.$$

Hence from the Hölder inequality one infers

$$a(t)t^{m+1} \leq 2\int_\Omega (v-u)_+ f\, dV \leq ||f||_{L^q(\Omega)}||(v-u)_+||_{q'}.$$

Inserting this estimate into (2.3) we arrive at

$$t \leq c_1(p', q, ||f||_{L^q(\Omega)})[||f||_q||(v-u)_+||_{L^{q'}(\Omega)}t^{-m-1}]^{p/n}$$

and consequently

$$t \leq c_2(p', q, ||f||_{L^q(\Omega)})||(v-u)_+||_{L^{q'}(\Omega)}^{\frac{p}{n+p(m+1)}}.$$

□

Corollary 2.6. *The last theorem gives a priori L^∞ estimate for the solutions of the Hessian equation (0.3) with the right hand side in L^q and a fixed boundary condition.*

Indeed, we apply the theorem for the solution u of

$$(dd^c u)^m \wedge \omega^{n-m} = f\omega^n$$

with given continuous boundary data φ and for v, which is the maximal function in $\mathcal{SH}_m(\omega, \overline{\Omega})$ matching the boundary condition (it exists by [Bl1]). Then u is bounded by a constant depending on $\Omega, ||\varphi|| = ||v||$, and $||f||_q$ since $||(v-u)_+||_{L^{q'}(\Omega)}$ is bounded (Proposition 2.2).

Corollary 2.7. *The solutions of the Hessian equation with the right hand sides uniformly bounded in L^q $q > n/m$ and given continuous boundary data form an equicontinuous family.*

For the proof follow [K5] p. 35, which deals with the Monge-Ampère case.

Below we state yet another stability theorem which we shall need later. Given the estimates we have already proven its proof follows the arguments from [K1].

Theorem 2.8. *Let $q > n/m$. Consider $u, v \in \mathcal{SH}_m(\omega, \overline{\Omega})$ such that $\{u < v\} \neq \emptyset$ and*

$$(dd^c u)^m \wedge \omega^{n-m} = f\omega^n, \ (dd^c v)^m \wedge \omega^{n-m} = g\omega^n$$

for some $f, g \in L^q(\Omega, dV)$. Then

$$\sup_\Omega(v-u) \leq sup_{\partial\Omega}(v-u) + c(q, m, n, diam(\Omega))||f-g||_{L^q(\Omega)}^{1/m}.$$

Remark 2.9. *The analogous stability theorem for the real m-Hessian equation ($m < n/2$) can be found in [W1], Theorem 5.5 (see also [CW]). There the optimal exponent q is equal to $n/2m$.*

Next we obtain a theorem on the existence of weak, continuous solutions when $\omega = \beta$ and the right hand side is in L^q, $q > n/m$.

Theorem 2.10. *Let Ω be smoothly bounded (m-1)-pseudoconvex domain (as in Theorem 1.6). Then for $q > n/m$, $f \in L^q(\Omega, dV)$ and continuous φ on $\partial\Omega$ there exists $u \in \mathcal{SH}_m(\omega, \overline{\Omega})$ satisfying*

$$(dd^c u)^m \wedge \beta^{n-m} = f\beta^n$$

and $u = \varphi$ on $\partial\Omega$.

Proof. For smooth, positive f this is the result of Li [Li] (Theorem 1.6). With our assumptions we approximate f in $L^q(\Omega, dV)$ by smooth positive f_j and approximate uniformly φ by smooth φ_j . The solutions u_j corresponding to f_j, φ_j are equicontinuous and uniformly bounded (Corollaries 2.6, 2.7). Thus we can pick up a subsequence converging uniformly to some $u \in \mathcal{SH}_m(\omega, \overline{\Omega})$. By the convergence theorem u solves the equation.

□

Remark 2.11. *Observe that for $\omega = \beta$, the plurisubharmonic function $u(z) = log||z||$ has a m-Hessian density in L^p for any $p < n/m$ which shows that the exponent n/m is optimal.*

Equipped with the existence and stability of weak solutions we can also prove the weak Gårding inequality announced in Section 1:

Theorem 2.12. *Let $u_1, \cdots, u_m$ be locally bounded m-sh functions in some domain $\Omega \in \mathbb{C}^n$. Suppose $(dd^c u_j)^m \wedge \beta^{n-m} = f_j \beta^n$ for some nonnegative functions $f_j \in L^q(\Omega)$, $q > n/m$. Then*

$$dd^c u_1 \wedge \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m} \geq (f_1 \cdots f_m)^{\frac{1}{m}} \beta^n.$$

Proof. We can essentially follow the lines of the proof of the analogous result for psh functions from [K4] (see also [K5]). First observe that the inequality is purely local hence it suffices to prove it under the additional assumptions that Ω is a ball and all the functions u_i are defined in a slightly bigger ball. Hence one can use convolutions with smoothing kernel to produce a decreasing to u_i sequence of m-sh functions $\{u_{i,j}\}_{j=1}^{\infty}$ (cf. Proposition 1.3). Then given any collection of smooth positive functions $f_{i,k} \in L^q(\Omega)$, $q > n/m$ by [Li] we can solve the Dirichlet problems

$$\begin{cases} v_{i,j,k} \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^{\infty}(\Omega) \\ (dd^c v_{i,j,k})^m \wedge \beta^{n-m} = f_{i,k}\beta^n \\ v_{i,j,k}|_{\partial\Omega} = u_{i,j}. \end{cases}$$

For those smooth functions we can apply pointwise the Gårding inequality to conclude that

$$dd^c v_{1,j,k} \wedge \cdots \wedge dd^c v_{m,j,k} \wedge \beta^{n-m} \geq (f_{1,k} \cdots f_{m,k})^{\frac{1}{m}} \beta^n$$

for any $j,\ k \geq 1$. Then given any non negative $f_i \in L^q(\Omega)$, $q > n/m$ we can find an approximating sequence of smooth positive $\{f_{i,k}\}_{k=1}^{\infty}$ which converge in L^q to f_i. By the stability theorem the corresponding solutions $v_{i,j,k}$ (recall they the same boundary values $u_{i,j}$) converge uniformly as $k \to \infty$ to the m-sh functions $v_{i,j}$ (solving the limiting weak equation), and hence the inequality follows from

the continuity of Hessian currents under uniform convergence of their potentials. Now if we let $j \to \infty$ the boundary vaules decrease towards u_i and hence so do the functions $v_{i,j}$ by the comparison principle. The convergence is not uniform but monotonicity is still sufficient to guarantee the continuity and hence in the limit we obtain the claimed inequality. $\square$

Remark 2.13. *The weak Gårding inequality can be further generalized similarly to the $m = n$ case as in [D].*

3 L^∞ estimates and existence of weak solutions on compact Kähler manifolds

The a priori estimates from the previous section can be carried over to the case of compact Kähler manifolds as it was done in [K4] or [K5] for the Monge-Ampère equation. Let us consider a compact n-dimensional Kähler manifold X equipped with the fundamental form ω and recall that a continuous function u is $\omega - m$-subharmonic (shortly: $\omega - m$-sh) on X if

$$(\omega + dd^c u)^k \wedge \omega^{n-k} \geq 0, \quad k = 1, 2, ..., m.$$

The set of such functions is denoted by $\mathcal{SH}_m(X, \omega)$. We study the complex m-Hessian equation

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = f\omega^n \tag{3.1}$$

with given nonnegative function $f \in L^1(M)$, which is normalized by the condition

$$\int_X f\, \omega^n = \int_X \omega^n.$$

The solution is required to be $\omega - m$-sh. By the result of Hou [Hou] the solutions of the equation, for smooth positive f, exist on manifolds with nonnegative holomorphic bisectional curvature. In that case our a priori estimates will also give the existence of weak solutions for $f \geq 0$ in L^q, $q > n/m$.

We define for a compact set $K \subset X$ its capacity

$$cap_m(K) = \sup\{\int_K (\omega + dd^c u)^m \wedge \omega^{n-m} : u \in \mathcal{SH}_m(X, \omega), 0 \leq u \leq 1\}.$$

To use the local results we need also a capacity defined as follows. Let us consider two finite coverings by strictly pseudoconvex sets $\{B_s\}$, $\{B'_s\}$, $s = 1, 2, ..., N$ of X such that $\bar{B}'_s \subset B_s$ and in each B_s there exists $v_s \in PSH(B_s)$ with $dd^c v_s = \omega$ and $v_s = 0$ on ∂B_s. Given a compact set $K \subset X$ define $K_s = K \cap \overline{B'_s}$. Set

$$cap'_m(K) = \sum_s cap(K_s, B_s),$$

where $cap_m(K, B)$ denotes the relative capacity from the previous section. As in [K4] one can show that $cap_m(K)$ is comparable with $cap'_m(K)$: There exists $C > 0$ such that

$$\frac{1}{C} cap_m(K) \leq cap'_m(K) \leq C cap_m(K).$$

Hence, by Proposition 2.1 we have

$$V(K) \leq C(p, X) cap_m^p(K),$$

for $p < \frac{n}{n-m}$ and V the volume measured by ω^n.

With this estimate at our disposal we can obtain the same a priori estimates as in domains in $\mathbb{C}^n$. The proofs are almost identical. In the compact setting one has to make sure that instead of just a sum of m-sh functions one considers a convex combination of $\omega - m$-sh functions (see [K5]). In particular the following theorems hold.

Theorem 3.1. *Consider $q > n/m$, its conjugate q' and $p' \in (q', n/(n-m))$. Write $p = p'/q' > 1$. Consider $u, v \in \mathcal{SH}_m(X, \omega)$ such that $\{u < v\} \neq \emptyset$ and*

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = f\omega^n$$

for some $f \in L^q(dV)$. Then

$$\sup(v - u) \leq c(p', q, ||f||_{L^q(X)})||(v-u)_+||_{q'}^{\frac{p}{n+p(m+1)}}, \quad (v-u)_+ := \max(v-u, 0).$$

Corollary 3.2. *The family of solutions of the Hessian equation (3.1) with the right hand sides uniformly bounded in L^q $q > n/m$ are equicontinuous.*

Applying the theorem of Hou [Hou] and the above statements one immediately gets the following existence theorem.

Theorem 3.3. *Let X be a compact Kähler manifold with nonnegative holomorphic bisectional curvature. Then for $q > n/m$ and $f \in L^q(dV)$ there exists a unique function $u \in \mathcal{SH}_m(X, \omega)$ satisfying*

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = f\omega^n$$

and $\max u = 0$.

4 Local regularity

In this section we prove a counterpart of the main result in Błocki-Dinew [BD], where the case of the Monge-Ampère equation was studied. We shall treat only the $\omega = \beta$ case and use PDE notation (with σ_m defined in Introduction).

Theorem 4.1. *Assume that $n \geq 2$ and $p > n(m-1)$. Let $u \in W^{2,p}(\Omega)$, where Ω is a domain in $\mathbb{C}^n$, be a m-subharmonic solution of*

$$\sigma_m(u_{z_j \bar{z}_k}) = \psi > 0. \tag{4.1}$$

Assume that $\psi \in C^{1,1}(\Omega)$. Then for $\Omega' \Subset \Omega$

$$\sup_{\Omega'} \Delta u \leq C,$$

where C is a constant depending only on n, m, p, $\operatorname{dist}(\Omega', \partial\Omega)$, *$\inf_\Omega \psi$, $\sup_\Omega \psi$, $||\psi||_{C^{1,1}(\Omega)}$ and $||\Delta u||_{L^p(\Omega)}$.*

Proof. By $C_1, C_2, \dots$ we will denote possibly different constants depending only on the required quantities. Without loss of generality we may assume that $\Omega = B$ is the unit ball in $\mathbb{C}^n$ and that u is defined in some neighborhood of $\bar{B}$. We will use the notation $u_j = u_{z_j}$, $u_{\bar{j}} = u_{\bar{z}_j}$ with the notable exception of $u_{(\epsilon)}$ which is defined below.

Let us define, following [BT1], the Laplacian approximating operator

$$T = T_\epsilon(u) = \frac{n+1}{\epsilon^2}(u_{(\epsilon)} - u),$$

where

$$u_{(\epsilon)}(z) = \frac{1}{V(B(z,\epsilon))}\int_{B(z,\epsilon)} u\, dV.$$

Since $T_\epsilon u \to \Delta u$ weakly as $\epsilon \to 0$, it is enough to show a uniform upper bound for T independent of ϵ. Observe that since u is subharmonic we have $T_\epsilon(u) \geq 0$.

Before we continue let us state two lemmas. The first one is classical.

Lemma 4.2. *Let $u \in W^{2,p}(\Omega)$ (Ω is a domain in $\mathbb{C}^n$) be a subharmonic function. Given any $\Omega' \Subset \Omega$ the operator $T_\epsilon(z)$ is well defined on Ω' for any sufficiently small $\epsilon > 0$. Furthermore*

$$||T_\epsilon||_{L^p(\Omega')} \to ||\Delta u||_{L^p(\Omega')}$$

in particular $||T_\epsilon||_{L^p(\Omega')}$ is uniformly bounded for all $0 < \epsilon < \epsilon_0$.

Lemma 4.3. *The function $T_\epsilon(u)(z)$ for any $\epsilon > 0$ satisfies the following subharmonicity condition:*

$$\frac{\partial \sigma_m\big(u_{j\bar{k}}\big)}{\partial u_{i\bar{j}}} T_{\epsilon, i\bar{j}} \geq -C_1,$$

where $\frac{\partial \sigma_m\left(u_{j\bar{k}}\right)}{\partial u_{i\bar{j}}}$ is the (i,j)-th (m-1)-cominor of the matrix $u_{i\bar{j}}(z)$ and C_0 is a constant dependent only on n, m, $\inf_\Omega \psi$, $\sup_\Omega \psi$, and $||\psi||_{C^{1,1}(\Omega)}$.

Proof. Observe that $u_{(\epsilon)}$ is a convex combination of m-subharmonic functions, hence it is m-subharmonic. Therefore one has the inequality

$$(dd^c u_{(\epsilon)})^m \wedge \omega^{n-m} \geq 0.$$

In fact following the lines of the same argument in [BT1] (where it was applied to the Monge-Ampère operator) one can prove the stronger inequality

$$(dd^c u_{(\epsilon)})^m \wedge \omega^{n-m} \geq ((\psi^{1/m})_{(\epsilon)})^m. \tag{4.2}$$

Indeed, for smooth u this is just a consequence of the concavity of $\sigma_m^{1/m}$. For nonsmooth solutions one can repeat the Goffman-Serrin formalism just as in [BT1].

Thus using the weak Gårding inequality (Theorem 2.12) one has

$$(dd^c u)^{m-1} \wedge dd^c u_{(\epsilon)} \wedge \omega^{n-m} \geq \psi^{(m-1)/m}(\psi^{1/m})_{(\epsilon)} dV.$$

Next, identifying (n,n) forms and their densities one gets, up to a multiplicative numerical constant $c_{n,m}$, the following string of inequalities

$$\begin{aligned}
&\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}T_{\epsilon,i\bar{j}} = c_{n,m}1/\epsilon^2 dd^c(u_{(\epsilon)}-u)\wedge(dd^c u)^{m-1}\wedge\omega^{n-m}\\
\geq& c_{n,m}1/\epsilon^2\psi^{(m-1)/m}((\psi^{1/m})_{(\epsilon)}-\psi^{1/m}) = c_{n,m}\psi^{(m-1)/m}T_\epsilon(\psi^{1/m}).
\end{aligned}$$

But ψ is a strictly positive $\mathcal{C}^{1,1}$ function hence $T_\epsilon(\psi^{1/m})\geq -C_1(||\psi||,||\psi^{1/m}||_{C^{1,1}})$. Combining all those inequalities we obtain the claimed estimate. $\square$

¿From now on we drop the indice ϵ in what follows. We will use the same calculations as in [BD] which in turn relied on [Tr2]. For some $\alpha,\beta\geq 2$ to be determined later set

$$w := \eta(T)^\alpha,$$

where

$$\eta(z) := (1-|z|^2)^\beta.$$

Then

$$w_i = \eta_i(T)^\alpha + \alpha\eta(T)^{\alpha-1}(T)_i$$

and

$$\begin{aligned}
\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}w_{i\bar{j}} =& \alpha\eta(T)^{\alpha-1}\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}(T)_{i\bar{j}} + \alpha(\alpha-1)\eta(T)^{\alpha-2}\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}(T)_i(T)_{\bar{j}}\\
&+2\alpha(T)^{\alpha-1}\mathrm{Re}\Big(\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}\eta_i(T)_{\bar{j}}\Big) + (T)^\alpha\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}\eta_{i\bar{j}}.
\end{aligned}$$

By Lemma 4.3 and the Schwarz inequality for $t>0$

$$\begin{aligned}
&\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}w_{i\bar{j}} \geq -C_1\alpha\eta(T)^{\alpha-1} + \alpha(\alpha-1)\eta(T)^{\alpha-2}\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}(T)_i(T)_{\bar{j}}\\
&-t\alpha(T)^{\alpha-1}\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}(T)_i(T)_{\bar{j}} - \frac{1}{t}\alpha(T)^{\alpha-1}\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}\eta_i\eta_{\bar{j}} + (T)^\alpha\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}\eta_{i\bar{j}}.
\end{aligned}$$

Therefore with $t=(\alpha-1)\eta/T$ we get

$$\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}w_{i\bar{j}} \geq -C_1\alpha\eta(T)^{\alpha-1} + (T)^\alpha\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}\left(\eta_{i\bar{j}} - \frac{\alpha}{\alpha-1}\frac{\eta_i\eta_{\bar{j}}}{\eta}\right).$$

We now have

$$\begin{aligned}
\eta_i &= -\beta z_i\eta^{1-1/\beta}\\
\eta_{i\bar{j}} &= -\beta\delta_{i\bar{j}}\eta^{1-1/\beta} + \beta(\beta-1)\bar{z}_i z_j\eta^{1-2/\beta},
\end{aligned}$$

and thus

$$\big|\eta_{i\bar{j}}\big|,\ \big|\frac{\eta_i\eta_{\bar{j}}}{\eta}\big| \leq C(\beta)\,\eta^{1-2/\beta}.$$

Coupling the above inequalities we get

$$\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}w_{i\bar{j}} \geq -C_2(T)^{\alpha-1} - C_3 w^{1-2/\beta}(T)^{2\alpha/\beta}\sum_{i,j}\big|\frac{\partial\sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}}\big|.$$

Fix q with $n/m < q < p/m(m-1)$ (by our assumption on p such a choice is possible). By Lemma 4.2 $||T||_p$ and $||\Delta u||_p$ are under control. By Calderon-Zygmund inequalities we control $||u_{i\bar{j}}||_p$ too. Observe that $\frac{\partial \sigma_m\left(u_{j\bar{k}}\right)}{\partial u_{i\bar{j}}}$ is a sum of of products of $m-1$ factors of the type $u_{i\bar{j}}$ and therefore $||\frac{\partial \sigma_m\left(u_{j\bar{k}}\right)}{\partial u_{i\bar{j}}}||_{p/(m-1)}$ is also under control. It follows that for

$$\alpha = 1 + \frac{p}{qm}, \quad \beta = 2\big(\frac{qm+p}{p-qm(m-1)}\big)$$

we have

$$||\big(\frac{\partial \sigma_m\left(u_{j\bar{k}}\right)}{\partial u_{i\bar{j}}} w_{i\bar{j}}\big)_-||_{qm} \leq C_3(1 + (\sup_B w)^{1-2/\beta}),$$

where $f_- := -\min(f, 0)$. By Theorem 2.10 we can find continuous m-subharmonic v vanishing on ∂B and such that

$$\sigma_m(v_{i\bar{j}}) = ((u^{i\bar{j}} w_{i\bar{j}})_-)^m.$$

Then the weak Gårding inequality yields

$$\begin{aligned}
&\frac{\partial \sigma_m\left(u_{j\bar{k}}\right)}{\partial u_{i\bar{j}}} v_{i\bar{j}} = c_{n,m}(dd^c u)^{m-1} \wedge dd^c v \wedge \omega^{n-m} \\
\geq & c_{n,m}(\sigma_m(u_{i\bar{j}})^{(m-1)/m}(\sigma_m(v_{i\bar{j}}))^{1/m} \geq 1/C_4\big(\frac{\partial \sigma_m\left(u_{j\bar{k}}\right)}{\partial u_{i\bar{j}}} w_{i\bar{j}}\big)_- \\
\geq & -\frac{1}{C_4} \frac{\partial \sigma_m\left(u_{j\bar{k}}\right)}{\partial u_{i\bar{j}}} w_{i\bar{j}}.
\end{aligned}$$

By maximum prinicple we obtain that $w \leq -C_4 v$, since this inequality holds on ∂B. Applying the stability theorem (Theorem 2.8), with $u = 0$, we get

$$\begin{aligned}
\sup_B w \leq C_4||v|| \leq C_5(||\sigma_m(v_{i\bar{j}})||_q^{1/m}) &= C_5||\big(\frac{\partial \sigma_m\left(u_{z_j \bar{z}_k}\right)}{\partial u_{i\bar{j}}} w_{i\bar{j}}\big)_-||_{qn} \\
&\leq C_6(1 + (\sup_B w)^{1-2/\beta}).
\end{aligned}$$

Therefore $w \leq C_7$ and thus

$$T^\alpha \leq \frac{C_7}{\eta}$$

which is the desired bound.

□

Remark 4.4. *The analogous reasoning can be applied also to the real m-Hessian equation (using Wang stability theorem and existence of weak solutions). It turns out that for $m < n/2$ the corresponding exponent in the $W^{2,p}$ Sobolev space is equal to $n(m-1)/2$. Observe that this improves the $m(n-1)/2$ exponent obtained by different methods by Urbas [U]. Whether this exponent is optimal is however still unclear and would require construction of suitable Pogorelov type Hessian examples.*

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